Stable Discrete-Time Path-Tracking Control for a WMR*

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Abstract

A discrete-time kinematic model of a two wheel differentially driven mobile robot is obtained by direct integration of its continuous-time kinematic model. The discrete-time model obtained is used to design two discrete-time linearizing control laws with different singular manifolds. These control laws are used to propose a commutation control scheme that solves the path-tracking control problem. A stability proof of the resulting control scheme is presented.

1 Introduction

The rise of the computational power in last decades has made most of the controllers digitally implemented and therefore they operate in discrete time. This produces that the discretization of continuous-time systems becomes especially important.

Most of the controllers for nonlinear systems are designed based on the continuous-time models producing continuous-time controllers. The main reason to use continuous-time models of nonlinear systems instead of their discrete-time models is due to the complexity to obtain an exact discrete-time model from the continuous-time one. In this work a discrete-time kinematic model of a class of wheeled mobile robot (WMR) called "two wheel differentially driven mobile robot" is obtained by direct integration of its continuous-time counterpart model.

One of the control alternatives for nonlinear systems is the control via feedback linearization. This is a control strategy with a complete theory developed by considering mainly the continuous-time case [1, 2]. For the study of nonlinear discrete-time systems, see [1], Chapter 14, or [3] where the subject is treated more deeply. Also, different notions of feedback linearization are studied in [4].

There are several works that solve the path-tracking problem of a WMR using differents control strategy are in particular, most of them are continuous-time controllers. As an example in [5, 6] the results of a mobile robot controlled by a nonlinear linearizing feedback are presented. In both cases the problem is solved by considering the continuous-time model of a two-wheel differentially driven mobile robot. In this work a discrete-time nonlinear control scheme is developed and its stability is analyzed. This control scheme solves the path-tracking problem based on the commutation of two discrete-time nonlinear controllers via feedback linearization.

The paper is organized as follows: In Section 2 the discretization of the continuous-time kinematic model of a two-wheel differentially driven mobile robot is obtained. In Section 3 the explicit design of a path-tracking control via feedback linearization is developed. In Section 4 a stability proof of the proposed control scheme is developed. Finally, the conclusions are presented in Section 5.

2 Discrete-time model

A *two-wheel differentially driven mobile robot* is shown in Figure 1. This class of WMR is driven by a motor in each wheel, a change of direction is obtained by the difference of velocity between the two wheels. Its continuous-time kinematic model is well known in the literature [7], and it is given by

$$\dot{x}_1 = u_1 \cos\theta_0 \quad (1a)$$
$$\dot{x}_2 = u_1 \sin\theta_0 \quad (1b)$$
$$\dot{\theta}_0 = u_2 \quad (1c)$$

where (x_1, x_2) represents the coordinates of the center of the axe of the wheels on the plane (X_1, X_2) and θ_0 is the angle that the longitudinal axis of the robot makes with the axis X_1. The input signal u_1 represents the linear velocity of the robot and u_2 its angular velocity.

To obtain the corresponding exact discrete-time model of system (1), consider a non-zero positive scalar constant sample time T and define the time interval between two consecutive sampling instants as,

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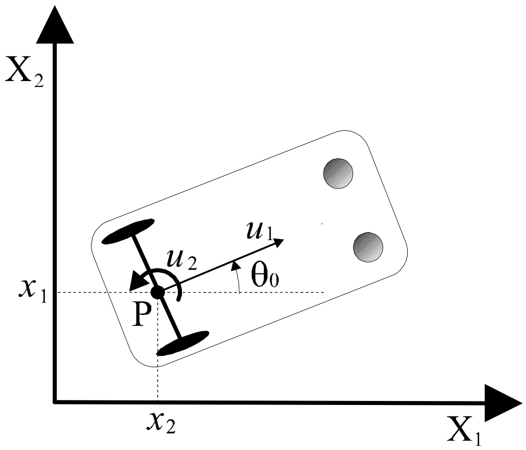

Figure 1: Two-wheel differentially driven mobile robot.

$$t_k = \{t \in \mathbb{R} \mid t \in [kT, (k+1)T), k = 0, 1, 2, \ldots\}.$$

Also, define $u_1(kT)$ and $u_2(kT)$ as the constant values that the input signals $u_1(t)$ and $u_2(t)$ take over the time intervals t_k for $k = 0, 1, 2, 3, \ldots$. For the sake of simplicity in the discrete-time case, the general notation $\zeta^+ = \zeta(kT+T)$ and $\zeta = \zeta(kT)$ will be used in the rest of the paper.

Considering first the direct integration of differential equation (1c),

$$\int_{kT}^{t} d\theta_0 = \int_{kT}^{t} u_2 \; dt$$

from where, it is obtained

$$\theta_0(t) = \theta_0 + (t - kT)\, u_2, \tag{2}$$

for $t \in t_k$. Integrating now equation (1a) with θ_0 given as in (2) produces,

$$\int_{kT}^{t} dx_1 = \int_{kT}^{t} u_1 \cos(\theta_0 + (t-kT)\, u_2)\, dt,$$

that leads to,

$$x_1(t) = x_1 + \frac{u_1}{u_2}[-\sin\theta_0 + \sin(\theta_0 + (t-kT)\, u_2)] \tag{3}$$

for $t \in t_k$. Finally, from equation (1b) with θ_0 as in (2),

$$\int_{kT}^{t} dx_2 = \int_{kT}^{t} u_1 \sin(\theta_0 + (t-kT)\, u_2)\, dt,$$

that, obtaining the integrals,

$$x_2(t) = x_2 + \frac{u_1}{u_2}[\cos\theta_0 - \cos(\theta_0 + (t-kT)\, u_2)] \tag{4}$$

for $t \in t_k$. In order to obtain the desired discrete-time model the equations (2), (3) and (4) are evaluated at the end of the interval t_k, this is,

$$\begin{aligned} x_1^+ &= x_1 + \frac{u_1}{u_2}[-\sin\theta_0 + \sin(\theta_0 + Tu_2)] \\ x_2^+ &= x_2 + \frac{u_1}{u_2}[\cos\theta_0 - \cos(\theta_0 + Tu_2)] \\ \theta_0^+ &= \theta_0 + Tu_2. \end{aligned} \tag{5}$$

By means of simple manipulations, system (5) can be rewritten as,

$$\begin{aligned} x_1^+ &= x_1 + 2u_1\psi\cos\gamma \\ x_2^+ &= x_2 + 2u_1\psi\sin\gamma \\ \theta_0^+ &= \theta_0 + u_2 T. \end{aligned} \tag{6}$$

where

$$\psi = \begin{cases} \frac{\sin\left(\frac{T}{2}u_2\right)}{u_2} & \text{if } u_2 \neq 0 \\ \frac{T}{2} & \text{if } u_2 = 0 \end{cases} \quad \text{and } \gamma = \theta_0 + \frac{T}{2}u_2.$$

Remark 2.1 *Note that,*

$$\lim_{u_2 \to 0} \frac{\sin\left(\frac{T}{2}u_2\right)}{u_2} = \frac{T}{2},$$

then the function ψ is continuous. In particular, when $u_2 = 0$ the state θ_0 remain constant accordingly to the property of the continuous-time model (1). It is easy to verify that the discrete-time model obtained from (1) when $u_2 = 0$ correspond to the discrete-time model (6) with $\psi(0) = \frac{T}{2}$. On the other hand the function ψ is equal to zero when $|T\xi_2| = 2m\pi$ for $m = 1, 2, \ldots$.

3 Discrete-time control scheme

This section is devoted to the design of a discrete-time controller that solve the path-tracking problem for system (6). This control strategy is based on the commutation between two discrete-time linearizing feedbacks. One of them considers an output function that fully linearize the system. The other linearizing control law considers an output function that allow to linearize the input-output response with one dimensional stable internal dynamic. The later controller works as an alternative control when the first one is in the neighborhood of its respective singular manifold.

The output function $h(x) = [x_1, x_2]^T$ fully linearize system (1) and most of its continuous-time controllers via feedback linearization are based on it. However, the consideration of the same output function to design a discrete-time controller for the system (6) does not produce the same result, achieving instead, an input-output linearization with unstable internal dynamics.

Consider now, for the discrete-time counterpart, the output function, $\bar{h}(x) = \left[\bar{h}_1(x), \bar{h}_2(x)\right]^T$,

$$\begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix} = \begin{bmatrix} x_1 \sin\gamma^- - x_2\cos\gamma^- \\ \theta_0^- \end{bmatrix} \tag{7}$$

where $\gamma^- = \frac{\theta_0+\theta_0^-}{2}$. In the above equation and for the rest of the paper, the notation $\zeta^- = \zeta(kT-T)$ will be adopted.

Motivated by the output function (7) it is proposed the dynamic extension,

$$\begin{aligned} \xi_1 &= \theta_0^-, \quad \xi_1^+ = \theta_0 \\ \xi_2 &= u_2, \quad \xi_2^+ = w_2 \\ w_1 &= u_1, \end{aligned} \tag{8}$$

that allow to rewrite (6) as the extended system,

$$x^+ = f(x, w) \tag{9}$$

where,

$$\begin{aligned} x &= [x_1, x_2, \xi_1, \theta_0, \xi_2]^T \\ w &= [w_1, w_2]^T \\ f(x,w) &= \begin{bmatrix} x_1 + 2\psi w_1 \cos\gamma \\ x_2 + 2\psi w_1 \sin\gamma \\ \theta_0 \\ \theta_0 + T\xi_2 \\ w_2 \end{bmatrix}. \end{aligned}$$

Remark 3.1 *In plain words, the dynamic extension (8) amounts to add one pure time delay in front of the input u_2, and to store the value at the previous time instant of the variable θ_0, in order to sinthetize the control law.*

3.1 Design of a full linearizing control law

Next, the development of the full linearizing control law is presented by considering the augmented system (9) and the output function (7). Following the notation used in [1, 4], it is possible to compute for the component $\bar{h}_1$,

$$\begin{aligned} \bar{h}_1^0(x) &= \bar{h}_1(x) = x_1 \sin\gamma^- - x_2 \cos\gamma^- \\ \bar{h}_1^1(x) &= \bar{h}_1^0(x^+) = x_1 \sin\gamma - x_2 \cos\gamma \\ \bar{h}_1^2(x,w) &= \bar{h}_1^1(x^+) = x_1 \sin\gamma^+ - x_2 \cos\gamma^+ \\ &\quad -2\psi w_1 \sin(\gamma - \gamma^+) \end{aligned}$$

where $\gamma^+ = \theta_0 + T\xi_2 + \frac{T}{2} w_2$. For the output component $\bar{h}_2$ it is possible to compute,

$$\begin{aligned} \bar{h}_2^0(x) &= \bar{h}_2(x) = \xi_1 \\ \bar{h}_2^1(x) &= \bar{h}_2^0(x^+) = \theta_0 \\ \bar{h}_2^2(x) &= \bar{h}_2^1(x^+) = \theta_0 + T\xi_2 \\ \bar{h}_2^3(x,w) &= \bar{h}_2^2(x^+) = \theta_0 + T\xi_2 + Tw_2. \end{aligned}$$

From the last developments it is possible to write,

$$\begin{aligned} \begin{bmatrix} \bar{h}_1^2 \\ \bar{h}_2^3 \end{bmatrix} &= \begin{bmatrix} x_1 \sin\gamma^+ - x_2 \cos\gamma^+ \\ \theta_0 + T\xi_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} -2\psi \sin(\gamma - \gamma^+) & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \end{aligned}$$

where it is clear that the decoupling matrix

$$D = \begin{bmatrix} -2\psi \sin(\gamma - \gamma^+) & 0 \\ 0 & T \end{bmatrix}$$

is singular on the manifold,

$$\bar{S} = \left\{ (x,\xi) \in \mathbb{R}^5 \mid \sin\left(\frac{\theta_0 - \bar{v}_2}{2}\right) = 0 \text{ or } \psi = 0 \right\}. \tag{10}$$

A control feedback law that solves the path tracking problem is given by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{x_1 \sin\bar{\phi} - x_2 \cos\bar{\phi} - \bar{v}_1}{2\psi \sin\left(\frac{\theta_0 - \bar{v}_2}{2}\right)} \\ \dfrac{\bar{v}_2 - \theta_0 - T\xi_2}{T} \end{bmatrix} \tag{11}$$

where $\bar{\phi} = \frac{\bar{v}_2 + \theta_0 + T\xi_2}{2}$ and the new input variables $\bar{v}_1$ and $\bar{v}_2$ are,

$$\begin{aligned} \bar{v}_1 &= \bar{y}_{1d}^{[2]} + \bar{k}_1 \bar{e}_1^+ - \bar{k}_0 \bar{e}_1 \\ \bar{v}_2 &= \bar{y}_{2d}^{[3]} - \bar{m}_2 \bar{e}_2^{[2]} + \bar{m}_1 \bar{e}_2^+ - \bar{m}_0 \bar{e}_2 \end{aligned}$$

with the output error $\bar{e} = \bar{y} - \bar{y}_d$. In order to simplify the notation, it will be considered $\zeta^{[i]} = \zeta(kT + iT)$, $\zeta^{[-i]} = \zeta(kT - iT)$.

For the sake of conciseness the control law (11) designed for system (9), will be referred in the rest of the paper as $\bar{w}$.

Remark 3.2 *The sum of the scalar relative degrees of system (9)-(7) is equal to the dimension of (9), then feedback law (11) fully linearizes the system.*

Remark 3.3 *The control law $\bar{w}$ is not defined when the state belong to the singular manifold $\bar{S}$ given in (10).*

Remark 3.4 *The closed-loop system composed by the extended system (9) and the control law $\bar{w}$ produces linear dynamic of the errors $\bar{e}_1$ and $\bar{e}_2$ governed by*

$$\begin{aligned} \bar{e}_1^{[2]} - \bar{k}_1 \bar{e}_1^+ + \bar{k}_0 \bar{e}_1 = 0 \\ \bar{e}_2^{[3]} + \bar{m}_2 \bar{e}_2^{[2]} - \bar{m}_1 \bar{e}_2^+ + \bar{m}_0 \bar{e}_2 = 0. \end{aligned} \tag{12}$$

3.2 Complementary feedback law

The control law $\bar{w}$ is not defined on $\bar{S}$, in order to obtain a global path-tracking control scheme another control law with a different singular manifold could be designed as an alternative. This control could be enabled when the state is in the neighborhood of $\bar{S}$. To obtain this alternative control law, consider now, the output function

$$\tilde{h}(x) = \begin{bmatrix} \tilde{h}_1 \\ \tilde{h}_2 \end{bmatrix} = \begin{bmatrix} x_1 \cos\gamma^- + x_2 \sin\gamma^- \\ \xi_1 \end{bmatrix}. \tag{13}$$

As in the previous section,

$$\begin{bmatrix} \tilde{h}_1^1 \\ \tilde{h}_2^3 \end{bmatrix} = \begin{bmatrix} x_1\cos\gamma + x_2\sin\gamma \\ \theta_0 + T\xi_2 \end{bmatrix} + + \begin{bmatrix} 2\psi & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

In this case, the decoupling matrix

$$\tilde{D} = \begin{bmatrix} 2\psi & 0 \\ 0 & T \end{bmatrix}$$

is singular on the manifold

$$\tilde{S} = \left\{(x,\xi) \in \mathbb{R}^5 \mid \psi = 0\right\}. \tag{14}$$

When the states are not in the manifold $\tilde{S}$ it is possible to synthesize the feedback law,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\tilde{v}_1 - x_1\cos\gamma - x_2\sin\gamma}{2\psi} \\ \dfrac{\tilde{v}_2 - \theta_0 - T\xi_2}{T} \end{bmatrix} \tag{15}$$

where $\tilde{v}_1$ and $\tilde{v}_2$ are given as,

$$\begin{aligned} \tilde{v}_1 &= \tilde{y}_{1d}^{+} + \tilde{k}_0\tilde{e}_1 \\ \tilde{v}_2 &= \tilde{y}_{2d}^{[3]} - \tilde{m}_2\tilde{e}_2^{[2]} + \tilde{m}_1\tilde{e}_2^{+} - \tilde{m}_0\tilde{e}_2 \end{aligned}$$

with the output error $\tilde{e} = \tilde{y} - \tilde{y}_d$.

Again, for sake of conciseness the control law (15) design for system (9) will be referred in the rest of the paper as $\tilde{w}$.

Remark 3.5 *The sum of the scalar relative degrees of system (9)-(13) is equal to four, therefore the augmented system (9) posses a zero dynamic of dimension one.*

Remark 3.6 *The control law $\tilde{w}$ is not defined when the states belong to the singular manifold $\tilde{S}$ given in (14).*

Remark 3.7 *The closed-loop system composed by the extended system (9) and the control law $\tilde{w}$ produces linear dynamic of the errors $\tilde{e}_1$ and $\tilde{e}_2$ governed by*

$$\begin{gathered} \tilde{e}_1^{+} - k_0\tilde{e}_1 = 0 \\ \tilde{e}_2^{[3]} + \tilde{m}_2\tilde{e}_2^{[2]} - \tilde{m}_1\tilde{e}_2^{+} + \tilde{m}_0\tilde{e}_2 = 0. \end{gathered} \tag{16}$$

3.3 Commutation strategy

Next, a control scheme based on the commutation between control laws $\bar{w}$ and $\tilde{w}$ is proposed. The singularities on the feedback (15) ($\psi = 0$) appears when the angular velocity of the robot ξ_2 is equal to a multiple of the sampling frequency (14), i.e., $\xi_2 = \frac{2m\pi}{T}$. For most of the practical discrete-time applications the sampling period ($T \approx 0.1$s), is so that the singularities can only appear for very high angular velocities of the WMR. Therefore, it can be reasonably assumed that this singularity does not appears in practical robots. From the above explanation it is proposed a commutation strategy that mainly takes into account the singularities produced by $\bar{S}$ instead of $\tilde{S}$,

$$w = \begin{cases} \bar{w} & \text{if } \left|\sin\left(\frac{\theta_0 - \bar{v}_2}{2}\right)\right| \geq \varepsilon \\ \tilde{w} & \text{if } \left|\sin\left(\frac{\theta_0 - \bar{v}_2}{2}\right)\right| < \varepsilon \end{cases} \tag{17}$$

where ε is a small positive number that represent the threshold of commutation.

4 Stability analysis

This section is devoted to the analysis of the stability the closed-loop system composed by the extended system (9) and the control scheme (17) developed in the previous section.

The control scheme (17) is based on the commutation between the linearizing control laws $\bar{w}$ and $\tilde{w}$, therefore one of them must be enabled. The commutation scheme is designed in such a way that the control $\tilde{w}$ is enabled when the trajectories of the closed-loop system is in the neighborhood of the singular manifold $\bar{S}$, when this happens, it is possible to proof that the internal dynamic noncontrolled by $\tilde{w}$ is stable. In other hand, when $\bar{w}$ the internal dynamics vanish and then the closed-loop system (9)-(17) is asymptotically stable.

Combining the results announced above, it is possible to establish sufficient conditions for the stability of the closed-loop system (9)-(17).

In order to state the main result of this section, consider the following set of assumptions:

Assumption 4.1 *All the coefficients $\bar{k}_i$, $\tilde{k}_i$, $\bar{m}_i$ and $\tilde{m}_i$ are such that the roots of the polynomials (12) and (16) are positive reals smaller than one.*

Assumption 4.2 *The trajectories of the closed-loop system (9)-(17) produce a finite number of switching of the control scheme (17).*

Assumption 4.3 *The function $r_k = \sqrt{\bar{y}_{1d}^2 + \bar{y}_{1d}^2}$ belongs to the space $P^n = \{p_k \mid \|p_k\| \leq a + bk^n\}$, where a and b are positive reals.*

Remark 4.4 *Note that the Assumption 4.2 implies that exists a positive number k_0 such that for all $k \geq k_0$ one of the control laws $\bar{w}$ or $\tilde{w}$ remains enabled.*

Remark 4.5 *The function r_k represents the radius of the desired trajectory, i.e., it is equal to $\sqrt{x_{1d}^2 + x_{2d}^2}$.*

Remark 4.6 *Note that the Assumption 4.3 is far from been restrictive because the function r_k is bounded by an unbounded polynomial.*

In the rest of the section, $\|\cdot\|$ denotes the usual euclidean norm, and by abuse of notation it also denotes the corresponding induced matrix norm.

Theorem 4.7 *Consider the closed-loop system composed by the extended system (9) and the commutation scheme (17). Suppose that Assumptions 4.1, 4.2 and 4.3 are satisfied. Then, one of the following assertions does hold:*

1. *The error $\tilde{e}$ converges to zero and the error $\bar{e}_1$ is bounded by a suitable constant, and then, the system is stable.*

2. *The error $\bar{e}$ converge to zero and then the system is asymptotically stable.*

The Proof of Theorem 4.7 is broken down into a series of technical Lemmas.

Lemma 4.8 *The coordinate transformation defined as*

$$\hat{x} = \Psi(x) = \begin{bmatrix} \xi_1 \\ \theta_0 \\ \theta_0 + T\xi_2 \\ x_1\cos\gamma + x_2\sin\gamma \\ x_1\sin\gamma - x_2\cos\gamma \end{bmatrix}$$

where $\gamma = \theta_0 + \frac{T}{2}\xi_2$ and $\hat{x} = \left[\tilde{y}_2, \tilde{y}_2^+, \tilde{y}_2^{[2]}, \tilde{y}_1, \bar{y}_1\right]^T$, is globally invertible.
Proof. *Note that the determinant of the Jacobian matrix of $\Psi(x)$ is different to zero for all $x \in \mathbb{R}^5$.* ■

Remark 4.9 *The vector $\hat{x}$ is composed by the states controlled by the closed-loop system (9)-(15) and the state $\bar{y}_1$. Therefore, $\bar{y}_1$ can be considered as an internal noncontrolled dynamic of the linearizing control law $\tilde{w}$.*

Lemma 4.10 *Consider the closed-loop system composed by the extended system (9) and the commutation scheme (17). Suppose that the control law $\tilde{w}$ is enabled for all $k \geq 0$. Then, the tracking error $\tilde{e}$ converges asymptotically to zero, and the error $\bar{e}_1$ are bounded by a suitable constant.*

Proof. Defining error variables as, $z = \hat{x} - \hat{x}_d = \Psi(x) - \Psi(x_d)$, the extended model (9) can be rewrite in z-coordinates as

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_3 \\ z_3^+ &= -\tilde{m}_2 z_3 + \tilde{m}_1 z_2 - \tilde{m}_0 z_1 \\ z_4^+ &= \tilde{k}_0 z_4 \\ z_5^+ &= z_5\cos(z_\phi + \phi_d) + z_4\sin(z_\phi + \phi_d) \\ &\quad +2r_k \sin\left(\frac{z_\phi}{2} + \phi_d + \tan^{-1}\frac{\tilde{y}_{1d}}{\bar{y}_{1d}}\right)\sin\frac{z_\phi}{2}. \end{aligned} \tag{18}$$

where $\phi_d = \frac{\tilde{y}_{2d}^{[2]} - \tilde{y}_{2d}}{2}$, $z_\phi = \frac{z_3 - z_1}{2}$ and $r_k = \sqrt{\tilde{y}_{1d}^2 + \bar{y}_{1d}^2}$.

Consider the linear subsystems,

$$\hat{z}^+ = \tilde{A}\hat{z} \tag{19}$$

$$z_4^+ = \tilde{k}_0 z_4 \tag{20}$$

where

$$\begin{aligned} \hat{z} &= [z_1, z_2, z_3]^T \\ \tilde{A} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \tilde{m}_2 & -\tilde{m}_1 & \tilde{m}_0 \end{bmatrix}. \end{aligned}$$

The solution of (20) is given as

$$z_4 = \tilde{k}_0^k z_4(0). \tag{21}$$

Under Assumption 4.1 the solution of the subsystem (19) are bounded by

$$\|\hat{z}\| \leq C_0 \lambda_1^k \tag{22}$$

with

$$\begin{aligned} C_0 &= \|P\| \left\|P^{-1}\hat{z}(0)\right\| \\ P &= \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \end{aligned}$$

where λ_i are the eigenvalues of $\tilde{A}$ and $\lambda_1 > \lambda_2 > \lambda_3$. The details developments of constant C_0 is referred to [8]

From (21) and (22) it can be conclude that the error $\tilde{e}$ converges to zero.

Considering now the norm of the error variable z_ϕ, applying the triangle inequality and using (22), it is obtained

$$\|z_\phi\| \leq C_0 \lambda_1^k. \tag{23}$$

From the nonlinear subsystem in (18), the state equation of z_5 can be rewrite as

$$z_5^+ = g_k z_5 + f_k \tag{24}$$

where

$$\begin{aligned} g_k &= \cos(z_\phi + \phi_d) \\ f_k &= z_4 \sin(z_\phi + \phi_d) \\ &\quad +2r_k \sin\left(\frac{z_\phi}{2} + \phi_d + \tan^{-1}\frac{\tilde{y}_{1d}}{\bar{y}_{1d}}\right)\sin\frac{z_\phi}{2}. \end{aligned}$$

The solution of (24) is given as,

$$z_5 = \left(\prod_{i=0}^{k-1} g_i\right) z_5(0) + \sum_{i=0}^{k-2}\left(\prod_{j=i+1}^{k-1} g_j\right) f_i + f_{k-1}. \tag{25}$$

Since $\|g_k\| \leq 1$, then

$$\left\|\prod_{i=n}^{m} g_i\right\| \leq 1$$

and therefore the norm of z_5 is bounded by

$$\|z_5\| \leq \|z_5(0)\| + \sum_{i=0}^{k-1}\|f_i\|. \tag{26}$$

Now considering the inequalities (21) and (23), the facts that $\|\sin(\alpha)\| \leq 1$ and $\left\|\sin\left(\frac{z_\phi}{2}\right)\right\| \leq \left\|\frac{z_\phi}{2}\right\|$ and the Assumptiom 4.3, the norm of the second term of (26) is bounded by

$$\sum_{i=0}^{k-1} \|f_i\| \leq \|z_4(0)\| \sum_{i=0}^{k-1} \tilde{k}_0^i + aC_0 \sum_{i=0}^{k-1} \lambda_1^i + bC_0 \sum_{i=0}^{k-1} i^n \lambda_1^i.$$

Under Assumption 4.1, it is a well-known result in calculus that

$$\sum_{i=0}^{k-1} \tilde{k}_0^i \leq \frac{1}{1-\tilde{k}_0}$$
$$\sum_{i=0}^{k-1} \lambda_1^i \leq \frac{1}{1-\lambda_1}$$

and by mean of the ratio test of convergence it is true that

$$\sum_{i=0}^{k-1} i^n \lambda_1^i \leq C_1.$$

Therefore, the solution z_5 is bounded by

$$\|z_5\| \leq \|z_5(0)\| + \frac{\|z_4(0)\|}{1-\tilde{k}_0} + \frac{aC_0}{1-\lambda_1} + bC_0C_1,$$

and hence the error $\bar{e}_1$ is bounded. ■

Lemma 4.11 *Consider the closed-loop system composed by the extended system (9) and the commutation scheme (17). Suppose that the control law $\bar{w}$ is enabled for all $t \geq 0$. Then the tracking error $\bar{e}$ converges asymptotically to zero.*

Proof. From Remark 3.2 it is known that the zero dynamic is null, then the convergence of the tracking error $\bar{e}$ implies asymptotic convergence of all the errors of the complete system. ■

Proof of Theorem 4.7: Since the closed-loop system is time-invariant, Lemmas 4.10 and 4.11 can be applied after each commutation by shifting the time-axis. This implies that the closed-loop system does not have finite escape time. Since only a finite number of commutations occur, one and only one of the assertions of Theorem 4.7 can arise. This concludes the Proof. ■

5 Conclusions

The discrete-time kinematic model of a two-wheel differentially driven mobile robot is obtained by direct integration of its continuous-time model. This discrete-time model is used to design a control scheme that solves the path tracking problem.

The output function usually used to obtain a fully linearizable control law for a two-wheel differentially driven mobile robot in continuous-time applications can not be longer applied to the discrete-time system. A new output function is proposed to obtain a fully linearizable discrete-time controller. This new discrete-time controller has a singular manifold that does not allow to solve the path tracking problem for any desired trajectory. In order to obtain a globally defined control scheme, it is designed an alternative linearizing control law with a different singular manifold that is enabled when the states are in the neighborhood of the singular manifold of the first one. The proposed commutation scheme that solves the path tracking problem is based on the commutation of these two discrete-time control laws.

The control scheme proposed has a stable behavior as is proved in this work. This proof shows the viability of its implementation. It is important to remark that the conditions established in the stability proof are rather strong, and far from being necessary.

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