Finite Inclusions Theorem based algorithm for Robust Stability of Time Delay Systems

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Abstract

A new version of the Finite Inclusions Theorem [6] for quasipolynomials of the retarded type is presented. The improvements consist of a reduced search interval for the frequencies and of new conditions which are easier to test. An algorithm for the determination of the robust stability of a polytopic family of quasipolynomials and an illustrative academic example are given.

1. Introduction

A difficulty in using two of the most important results of the frequency domain approach for stability analysis, the Nyquist Theorem and the Zero Exclusion Principle, is that an infinite number of frequencies must be tested. A remedy to this problem is given by the *Finite Nyquist Theorem* and the *Finite Inclusions Theorem* for polynomials [2] and the *Finite Zero Exclusion Principle* developed by A. Rantzer in [5]. These valuable semi-analytic tools show that it is possible to reduce the testing set to a finite number of frequencies. Both results were extended to the case of quasipolynomials with real and complex coefficients of the retarded type in [6], [7].

In this contribution, we improve the results given in [6] with a further reduction of the frequencies search interval on the imaginary axis and conditions that are more convenient to use. These new conditions give rise to an algorithm for testing the robust stability of polytopic families of quasipolynomials of the retarded type.

The main motivation of this work is the robustness analysis (see [3] for an historical review on robustness of quasipolynomials) of time delay systems [1] of retarded type described by

$$\dot{x}(t) = \sum_{i=0}^{q} A_i x(t-\tau_i)$$

where $A_i, i = 1, ..., q$ are real $n \times n$ matrices, and $0 = \tau_0 < \tau_1 < ... < \tau_q$ are time delays. The characteristic equation of this system is a quasipolynomial which can be written as

$$f(s) = \sum_{l=0}^{m} p_l(s) e^{\beta_l s} \qquad (1.1)$$

where $\beta_0 < \beta_1 < ... < \beta_m = 0$ are shifts which depend on the delays and where $p_k(s)$ are real polynomials of the form

$$p_l(s) = \alpha_{0l} + \alpha_{1l}s + \alpha_{2l}s^2 + \cdots + \alpha_{n-1,l}s^{n-1}$$
$$l = 0, ..., m-1$$

with $p_m(s) = \det(sI - A_0) = \alpha_{0m} + a_{1m}s + ... + s^n$.

Quasipolynomial $f(s)$ is of the retarded type, and has a finite number of roots in any right half complex plane. Here, $f(s)$ *is assumed to have no roots on the imaginary axis.*

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This note is organized as follows: the reduction of the search interval for the Nyquist Theorem for real quasipolynomials is explained in Section 2 and the new conditions for the Finite Nyquist Theorem are established in Section 3. In Section 4 the new version of the Finite Inclusions Theorem for the robust stability analysis of polytopic families of quasipolynomials is given. An algorithm for the robust stability analysis is presented in Section 5 and an illustrative example is given in Section 6. The contribution ends with some concluding remarks.

2. Nyquist Theorem for real quasipolynomials of retarded type

In [6] we proved Theorem **2.1** which states that the numbers of roots of (1.1) can be determined by the knowledge of the net change of the argument of $f(j\omega)$ on a finite segment of the imaginary axis. Here, we improve the result by reducing substantially this segment. To do so we first prove the following proposition.

Proposition 1. *Let ε be given and let ξ_ε be the smallest positive root of the equation*

$$C_0\xi^n + C_1\xi^{n-1}\cdots + C_{n-1}\xi = \sin\frac{\varepsilon}{2} \tag{2.1}$$

where

$$0 < C_k = \sum_{l=0}^{m} |a_{kl}|, \quad \begin{array}{l} k=0,1,...,n-1 \\ l=0,1,...,m \end{array} . \tag{2.2}$$

Then the change of the argument of $f(j\omega)$ on $[jR,\infty]$ for $R \geq \frac{1}{\xi_\varepsilon}$ is bounded by ε.

Proof. The quasipolynomial $f(s)$ can be written as

$$f(s) = s^n\left(1+\sum_{l=0}^{m}\sum_{k=0}^{n-1}\alpha_{kl}s^{k-n}e^{\beta_l s}\right),$$

so that its argument on the imaginary axis is

$$\arg(f(j\omega)) = n\arg(j\omega) + \arg(Z) \tag{2.3}$$

where

$$\begin{aligned} Z &= 1+z, \\ z &= \sum_{k=0}^{n-1}\sum_{l=0}^{m}\alpha_{kl}(j\omega)^{k-n}e^{j\beta_l\omega}. \end{aligned}$$

We observe that when $\omega\to\infty$, $\arg f(j\omega)\to n\pi$ because $\arg(Z)\to 0$, so we may fix an argument ε for Z in order to know the frequency for which $\arg f(j\omega)$ does not change for $\omega \geq R$. Note that for $\omega = R \gg 1$ the complex number Z belongs to the disk of radius $\delta(\varepsilon) = \sin\frac{\varepsilon}{2}$ centered at 1 depicted in Figure 2.1. In this manner we

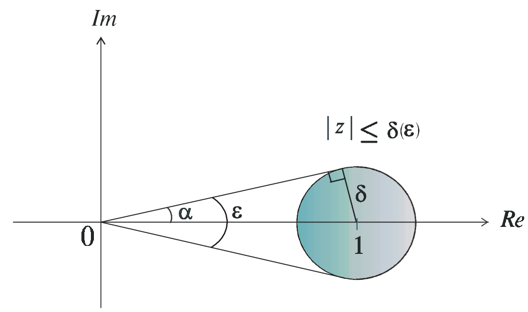

Figure 2.1: Z belongs to a disk of radius $\delta(\varepsilon)$

only have to bound z. Now, considering that

$$\begin{aligned} \left|e^{j\beta_l R}\right| &= 1, l=0,1,...,m, \\ \left|(jR)^{-k}\right| &= \left|R^{-k}\right|, k=1,2,..,n, \end{aligned}$$

and defining a new variable $R=\frac{1}{\xi}$, $\xi\in\mathcal{R}$; we observe that the module of z satisfies:

$$|z| \leq C_0\xi^n + C_1\xi^{n-1} + \cdots + C_{n-1}\xi$$

where C_k, $k=0,1,...,n-1$, is defined by (2.2).

According to Figure 2.2, we may consider $|z|$ as a positive curve which depends on ξ and it can be bounded by the line $\delta(\varepsilon) = \sin\frac{\varepsilon}{2}$, so the argument ε accumulated by $f(j\omega)$ does not change for $R \geq \frac{1}{\xi_\varepsilon}$. We note that the root ξ must be real because is the inverse of R, which is also real.

The new bound of Proposition 1 leads to the following results (proofs are similar to those in [6]).

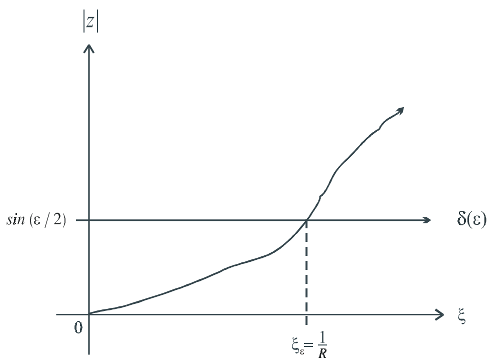

Figure 2.2: The curve $|z|$ is bounded by the line $\delta(\varepsilon)$.

Theorem 2.1. *[6] Let $0 \leq \varepsilon < \frac{\pi}{2}$ be given, and let $\Phi_{[0,jR]}$ be the net change of the argument of $f(j\omega)$ on the finite segment $[0, jR]$, with $R = \frac{1}{\xi_\varepsilon}$, where ξ_ε is the smallest positive root of (2.1) with C_k, $k = 0, 1, ..., n-1$ defined in (2.2). Then, the unique integer $N \geq 0$ that satisfies*

$$-\varepsilon \leq N\pi - \frac{1}{2}(n - \Phi_{[0,jR]}) \leq \varepsilon \tag{2.4}$$

is equal to the number of unstable roots of $f(s)$.

Corollary 1. *[6] Let $\Phi_{[0,jR]}$ be the contribution of the finite segment $[0, jR]$, $R = \frac{1}{\xi_{\frac{\pi}{2}}}$, to the net change of argument of the quasipolynomial $f(s)$ described by (1.1), here $\xi_{\frac{\pi}{2}}$ is the smallest positive root of (2.1) with C_k, $k = 0, 1, ..., n-1$ and $\varepsilon = \frac{\pi}{2}$ given by (2.2).*
Then $f(s)$ is stable if and only if

$$n\frac{\pi}{2} - \frac{\pi}{2} < \Phi_{[0,jR]} < n\frac{\pi}{2} + \frac{\pi}{2}. \tag{2.5}$$

3. Finite Nyquist Theorem for Real Quasipolynomials

In what follows, we provide conditions for the Finite Nyquist Theorem which are more convenient than those developed in [6]. Indeed, they give more insight on how the first and last frequencies should be selected.

Theorem 3.1. *The quasipolynomial $f(s)$ is stable if and only if there exist angles $\theta_i \in \mathbb{R}$ for $1 \leq i \leq r$ and real frequencies $0 \leq \omega_1 < \omega_2 < \cdots < \omega_r \leq R$ where R is defined in Corollary 1, such that*

$$-\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}, \tag{3.1}$$

$$n\frac{\pi}{2} - \pi < \theta_r < n\frac{\pi}{2} + \pi, \tag{3.2}$$

$$\forall\ 1 \leq i < r-1 \quad |\theta_{i+1} - \theta_i| \leq \pi, \tag{3.3}$$

$$\forall\ 1 \leq i \leq r : f(j\omega_i) \neq 0 \tag{3.4}$$

$$\forall\ 1 \leq i \leq r : \arg f(j\omega_i) \equiv \theta_i \pmod{2\pi}. \tag{3.5}$$

Proof. *We show that these conditions are equivalent to those of Theorem A.1 of the appendix which was proven in [6].*

It follows from Theorem A.1 that there exists a list of frequencies and arguments such that (A.1), (A.2) are satisfied. According to the proof in [6], we can assume with no loss of generality that $0 < \tilde{\theta}_{i+1} - \tilde{\theta}_i < \pi$, $1 \leq i < r-1$:

$$\begin{array}{ccccc} 0 \leq \tilde{\omega}_1 & < \tilde{\omega}_2 & ... & < \tilde{\omega}_{r-1} & < \tilde{\omega}_r, \\ \tilde{\theta}_1 < & \tilde{\theta}_2 < & ... & \tilde{\theta}_{r-1} < & \tilde{\theta}_r. \end{array} \tag{3.6}$$

Let $\theta_r = \tilde{\theta}_{\tilde{r}} - \tilde{\theta}_1$, and observe that $\theta_r \leq \tilde{\theta}_{\tilde{r}}$. Due to the continuity of $\arg f(j\omega)$ there exists $\omega_r < R$ such that $\theta_r = \arg f(j\omega_r)$ and there exists $\eta < r$ such that $\tilde{\theta}_\eta < \theta_r \leq \tilde{\theta}_{\eta+1}$.

A.2 implies that $0 < \theta_r - \tilde{\theta}_\eta < \pi$, therefore we can remove from list (3.6) the arguments $\left\{\tilde{\theta}_{\eta+1}, ..., \tilde{\theta}_{\tilde{r}-1}, \tilde{\theta}_{\tilde{r}}\right\}$ with their corresponding frequencies.
If $0 = \theta_1 \leq \tilde{\theta}_1 < \pi$, we must add θ_1 to the list.

But if $\tilde{\theta}_1 > \pi$, by continuity, there exist frequencies and corresponding arguments $\{\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_\zeta\}$ such that $\theta_1 < \hat{\theta}_1 < \hat{\theta}_2 < ... < \hat{\theta}_\zeta < \tilde{\theta}_1$ and such that all differences $0 < \hat{\theta}_{l+1} - \hat{\theta}_l < \pi$ for $1 < l < \zeta - 1$. Then *the arguments $\{\theta_1 = 0, \hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_\zeta\}$ are added to the list.*

The list obtained using this procedure satisfies all the conditions of Theorem 3.1.
Conversely, from Theorem 3.1 there exists a list of frequencies and arguments such that (3.1), (3.2) and (3.3) are satisfied. According to the proof in [6], we can assume with no loss of generality that $0 < \theta_{i+1} - \theta_i < \pi$, $1 \leq i < r-1$:

$$\begin{array}{ccccc} \omega_1 & < \omega_2 & ... & < \omega_{r-1} & < \omega_r \leq R, \\ \theta_1 < & \theta_2 < & ... & \theta_{r-1} < & \theta_r. \end{array} \quad (3.7)$$

Now, observe that $\tilde{\theta}_1 = 0$ satisfies $-\frac{\pi}{4} < \tilde{\theta}_1 < \frac{\pi}{4}$.
If $\tilde{\theta}_1 < \theta_1$, then it follows from (3.1) that $0 \leq \theta_1 - \tilde{\theta}_1 \leq \frac{3}{4}\pi$ so that $|\theta_1 - \tilde{\theta}_1| < \pi$ and we add $\tilde{\theta}_1$ to the list (3.7). If $\theta_1 = 0$ the lower part of the list is not modified.

By the continuity of the argument, there exists $\tilde{\theta}_{\tilde{r}}$ and $\tilde{\omega}_{\tilde{r}}$ such that

$$n\frac{\pi}{2} - \frac{\pi}{4} < \tilde{\theta}_{\tilde{r}} < n\frac{\pi}{2} + \frac{\pi}{4}. \quad (3.8)$$

If $\tilde{\theta}_{\tilde{r}} > \theta_r$ we observe from (3.8) and (3.2) that $-\frac{5}{4}\pi < \tilde{\theta}_{\tilde{r}} - \theta_r < \frac{5}{4}\pi$.
If $0 < \tilde{\theta}_{\tilde{r}} - \theta_r < \pi$ we add $\tilde{\theta}_{\tilde{r}}$ to the list.
If $\pi < \tilde{\theta}_{\tilde{r}} - \theta_r < \frac{5}{4}\pi$, by continuity of $\arg f(j\omega)$ there exists $\hat{\theta}_1$ such that $\theta_r < \hat{\theta}_1 < \tilde{\theta}_{\tilde{r}}$ and $\tilde{\theta}_{\tilde{r}} - \hat{\theta}_1 < \pi$ and $\hat{\theta}_1 - \theta_r < \pi$, and $\hat{\theta}_1$ is added to the list. If $\tilde{\theta}_{\tilde{r}} < \theta_r$, by continuity of the argument, there exists l such that $\theta_l < \tilde{\theta}_{\tilde{r}} \leq \theta_{l+1}$ in this case observe that $\left|\tilde{\theta}_{\tilde{r}} - \theta_l\right| < \pi$ then $\tilde{\theta}_{\tilde{r}}$ is added to the list and $\{\theta_{l+1}, ..., \theta_r\}$ is eliminated from the list.

The list obtained using this procedure satisfies all the conditions of Theorem A.1 ■

We note that condition (3.1) shows that the first argument must be between the first and the fourth quadrant of the complex plane, while condition (3.3) says that the change of the argument between two consecutive angles has no to be more than 180 degrees. Finally, condition (3.2) says that last argument has to be near $n\pi/2$.

Remark 1. *The minimum number of frequencies that are required in order to establish the stability of the quasipolynomial (1.1) is equal to $m+1$ if $n = 2m$ is even, and it is equal to $m+1$ if $n = 2m+1$ is odd.*

4. The Finite Inclusions Theorem and robust stability

The reduction of the frequencies search interval, and the improved conditions developed in the last section lead us to a new version of the Finite Inclusions Theorem [6] for the stability analysis of polytopic families of quasipolynomials of the retarded type. We consider polytopic family of quasipolynomials described by

$$F = \left\{ \sum_{u=1}^{T} \mu_u f_u(s) | \ \mu_u \geq 0, \sum_{u=1}^{T} \mu_u = 1 \right\}, \quad (4.1)$$

where $f_u(s)$ are quasipolynomials of the form (1.1).

Definition 1. *The family F is said to be robustly stable if all its members are stables.*

For a given frequency $\omega \in R$, the value set of F at this frequency is defined as

$$V_F(\omega) = \left\{ \sum_{u=1}^{T} \mu_u f_u(j\omega) | \ \mu_u \geq 0, \quad \sum_{u=1}^{T} \mu_u = 1 \right\}.$$

As for polynomials, the stability of the vertex quasipolynomials, $f_u(s)$, $u = 1, 2, ..., T$, does not imply that of the whole family (the stability of the edges is also required, see [4]). Observe that when using a graphical approach, the convexity of the polytopic family is inherited to the value set and, for each frequency ω the value set is the convex hull of the complex numbers $f_u(j\omega)$, $u = 1, 2, ..., T$. As a consequence, the testing set is reduced to the set of extreme quasipolynomials. Yet an infinite number of frequencies must still be tested. The following result allows a reduction of the number of frequencies to be checked.

Theorem 4.1. *The polytopic family defined in (4.1) is robustly stable if there exists $r \geq 1$,*
sectors $S_i = \{\rho e^{j\theta_i} \mid \rho > 0, \quad a_i < \theta_i < b_i\}$
for $1 \leq i \leq r$ and real frequencies
$0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_r \leq R$ such that

$$-\frac{\pi}{2} \quad \leq \quad a_1 < b_1 \leq \frac{\pi}{2}, \tag{4.2}$$

$$\frac{n\pi}{2} - \pi \quad \leq \quad a_r < b_r \leq \frac{n\pi}{2} + \pi, \tag{4.3}$$

$\forall \ 1 \leq i < r - 1 :$

$$\gamma = \max\{b_{i+1} - a_k, b_i - a_{i+1}\} \leq \pi, \tag{4.4}$$

$\forall \ 1 \leq u < T, \quad \forall 1 \leq i \leq r :$

$$f_u(j\omega_i) \in S_i = \{\rho e^{j\theta_i} \mid \rho > 0, a_i < \theta_i < b_i\}. \tag{4.5}$$

Proof. The proof follows closely that in [6].

Condition (4.2) says that the first sector, enclosing the first value set, must be between the first and the fourth quadrant of the complex plane and conditions (4.3) and (4.4) imply that two consecutive sectors have no to be more separated than 180 degrees and these sectors have to revolve a net change of $n\pi/2$ radians about the origin of the complex plane.

5. FIT Algorithm

In this section the algorithm for testing the robust stability of the family (4.1) is presented.

STEP 1. Verify the stability of the central quasipolynomial $f_0(s) = \frac{1}{T}\sum_{u=1}^{T} f_u(s)$. To do it, compute R_0 with the help of Proposition 1 and the change of the argument of $f_0(j\omega)$ for $\omega \in [0, R_0]$. An appropriate numerical step must be selected. If the accumulated argument is $n\frac{\pi}{2}$, the quasipolynomial under consideration is stable according to Corollary 1. If it is not possible to conclude on the stability of $f_0(s)$ the procedure stops. Otherwise, we proceed to Step 2.

STEP 2. Determine the frequencies ω_i, $0 \leq i \leq r$ for which the Mikhailov Graph of $f_0(j\omega)$, $\omega \in [0, jR]$, crosses the real and imaginary axis of the complex plane.

STEP 3. For each ω_i, $0 \leq i \leq r$, evaluate $f_u(j\omega_i)$, $u = 1, ..., T$ and choose a_i and b_i such that condition (4.5) is satisfied. The first and the last sector should satisfy conditions (4.2) and (4.3). Consecutive sectors should also satisfy condition (4.4) (notice that they can be overlapped).

STEP 3. If all the conditions of the Finite Inclusion Theorem are satisfied for the family F, we can ensure the robust stability of the polytopic quasipolynomial family. If not, compute the frequencies for which the Mikhailov Graph of $f_0(j\omega)$, $\omega \in [0, jR]$, crosses the bisectrixes, add these frequencies to the testing set and go back to step 3.

Remark 2. *The conditions of Theorem 4.1 are sufficient. They do not permit to conclude on the instability of a given family.*

6. Illustrative example

Example 1. *Consider the polytopic family of quasipolynomials*

$$G = \left\{ \sum_{u=1}^{4} \mu_u g_u(s) | \mu_u \geq 0, \sum \mu_u = 1 \right\} \tag{6.1}$$

where

$$\begin{aligned} g_1(s) &= (s^2 + 3s + 3.5)e^{-3s} \\ &\quad + (s^4 + 4s^3 + 15.5s^2 + 11.5s + 14), \\ g_2(s) &= (s^2 + 3.6s + 4)e^{-3s} \\ &\quad + (s^4 + 3s^3 + 16s^2 + 13s + 15), \\ g_3(s) &= (s^2 + 4s + 2.6)e^{-3s} \\ &\quad + (s^4 + 4.5s^3 + 17s^2 + 12s + 17), \\ g_4(s) &= (s^2 + 2s + 2)e^{-3s} \\ &\quad + (s^4 + 5s^3 + 15s^2 + 12s + 16). \end{aligned}$$

One can verify that the frequencies and sectors of Table 1, which were determined according to the algorithm described above, satisfy the conditions of the theorem, therefore the family G is robustly stable.

i	ω_i	Value Set Vs_i	a_i	b_i	γ
1	0	Vs_1	$-\frac{1}{3}\pi$	$\frac{1}{18}\pi$	
2	0.9735	Vs_2	$\frac{1}{9}\pi$	$\frac{2}{3}\pi$	π
3	1.8706	Vs_3	$\frac{9}{5}\pi$	$\frac{10}{9}\pi$	π
4	4.0016	Vs_4	π	$\frac{14}{9}\pi$	$\frac{1}{9}\pi$

Table 1

The value set of the quasipolynomial family (6.1) along with the sectors are depicted on figure(6.1), where an scaling was employed.

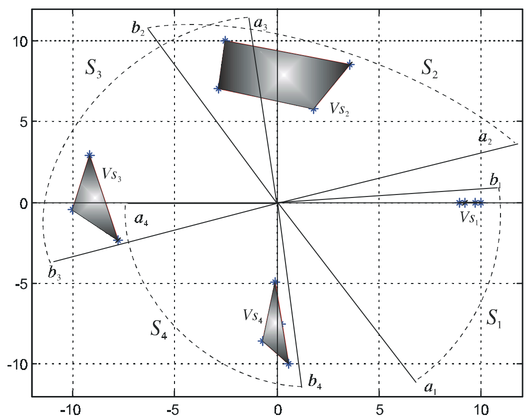

Figure 6.1: Value Set of G depicted at some frequencies

7. Concluding Remarks

In this contribution, the Finite Inclusions Theorem presented in [6] was improved by reducing the frequencies search interval and by providing easier testing conditions. This leads us to give an algorithm for the robust stability analysis of polytopic families of quasipolynomials applying the theorem.

Further work includes the development of an algorithm for testing the robust stability of interval quasipolynomials with real as well as complex coefficients and employing these results in a control framework.

A. The Finite Nyquist Theorem for quasipolynomials

Next we recall the Finite Inclusions Theorem for quasipolynomials which was proved in [6].

Theorem A.1. *[6] The Quasipolynomial $f(s)$ is stable if and only if there exists $\tilde{r} \geq 1$ angles $\tilde{\theta}_i \in \mathbb{R}$ for $1 \leq i \leq \tilde{r}$, and real numbers $0 \leq \tilde{\omega}_1 \leq \tilde{\omega}_2 \leq \cdots \leq \tilde{\omega}_r$ such that*

$$\begin{aligned} \frac{n\pi}{2} - \frac{\pi}{2} &< \tilde{\theta}_{\tilde{r}} - \tilde{\theta}_1 < \frac{n\pi}{2} + \frac{\pi}{2}, && \text{(A.1)} \\ \forall\, 1 &\leq i < \tilde{r} - 1 \quad \left|\tilde{\theta}_{i+1} - \tilde{\theta}_i\right| \leq \pi, && \text{(A.2)} \\ \forall\, 1 &\leq i \leq \tilde{r} : f(j\tilde{\omega}_i) \neq 0, \\ \forall\, 1 &\leq i \leq \tilde{r} : \arg f(j\tilde{\omega}_i) \equiv \tilde{\theta}_i \pmod{2\pi}. \end{aligned}$$

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