Time optimal control and the Potapov's Fundamental Matrix Inequality

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Introduction

The Time–Optimal Control (TOC) problem occupies a central place in optimal control theory, since time of movement is one of the most natural optimality criteria. In this work we show that the solution of the TOC problem can be given in terms of the solution of a classical power moment problem on a finite interval $[0,\theta]$ with help of the Fundamental Matrix Inequality (FMI) of Potapov [4]. An explicit solution of the optimal time θ (2.1) and the switching points of the optimal control $u(t)$ (2.3) are given.

Notations. We use $\mathbb{R}^n$,$\mathbb{C}$ to denote the sets of n–dimensional Euclidean space ($\mathbb{R}$ is the set of real numbers) and complex numbers, respectively. We will use $\mathcal{C}_L^f$ to denote the set of all functions $f : 0 \leq f(\tau) \leq L,\ \tau \in [a,b]$. The symbol $\mathcal{M}[a,b]$ stands for the set of all non-negative measures on $[a,b]$. $\bar{z}$ and w^* denote the complex conjugate of the number z and function w, respectively. The null space of a matrix A we denote by $ker A$.

Statement of the problem. We consider the time-optimal control (TOC) problem for the following system,

$$\dot{x} = Ax + bu,\ x(0) = x_0, |u| \leq 1 \qquad (0.1)$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (0.2)$$

Here $x \in \mathbb{R}^n$, obviously A is a $n \times n$ matrix. It is required to find the optimal control u such that $|u| \leq 1$, and the minimal θ–time such that $x(\theta) = 0$. This problem was considered in [1]. An analytical solution of this problem was given in [2] and [3] based on the treatment of an equivalent Markov power moment problem.

Now we write some notions about moment problems which are crucial for the present work:

L Markov moment problem in finite interval [a,b]

Let be given a sequence of real numbers $\{c_j\}_{j=0}^k$. Find the set of functions $f : f \in \mathcal{C}_L^f$ such that the relation

$$c_j = \int_a^b \tau^j f(\tau) d\tau,\ j \in \{0, \cdots, k\}. \qquad (0.3)$$

holds. We use $\mathcal{C}_L^f(\{c_j\}_{j=0}^k)$ to denote the set of solutions of (0.3). Rermark $\mathcal{C}_L^f(\{c_j\}_{j=0}^k) \subseteq \mathcal{C}_L^f$.

The finite Hausdorff moment problem. The classical power moment problem for an interval $[a,b]$ is stated as follows: Let be given a finite sequence of real numbers $\{s_j\}_{j=0}^k$, such that

$$s_j = \int_a^b \tau^j \sigma(d\tau),\ j \in \{0, \cdots, k\}. \qquad (0.4)$$

It is required to find the set of measures $\sigma : \sigma \in \mathcal{M}[a,b]$ such that (0.4) holds. We use $\mathcal{M}([a,b], \{s_j\}_{j=0}^k)$ to denote the set of solutions of (0.4). Remark that $\mathcal{M}([a,b], \{s_j\}_{j=0}^k) \subseteq \mathcal{M}[a,b]$.

Relation between the L–Markov power moment and the finite Hausdorff moment problem

The treatment of the L–Markov moment problem is usually connected with the problem of

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finding a holomorphic function, $z \in \mathbb{C} \setminus [a,b]$

$$c(z) = \int_a^b \frac{f(\tau)}{\tau - z} d\tau, \ f \in \mathcal{C}_L^f. \tag{0.5}$$

In terms of the asymptotic expansion

$$\begin{aligned} \int_a^b \frac{f(\tau)}{\tau - z} d\tau &= -\frac{1}{z} \int_a^b \left(1 - \frac{\tau}{z}\right)^{-1} f(\tau) d\tau \\ &= -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_a^b \tau^j f(\tau) d\tau \\ &= -\sum_{j=0}^{\infty} \frac{c_j^f}{z^{j+1}}, \end{aligned} \tag{0.6}$$

it is required to find the set of functions $f : f \in \mathcal{C}_L^f$ such that $c_j^f = c_j$, $j \in \{0, \ldots, k\}$, that is $f \in \mathcal{C}_L^f(\{c_j\}_{j=0}^k)$. Here $c_j^f = \int_a^b \tau^j f(\tau) d\tau$, $f \in \mathcal{C}_L^f$ and c_j is number of a given sequence of numbers $\{c_j\}_{j=0}^k$.
In a similar way, a holomorphic function defined in $z \in \mathbb{C} \setminus [a,b]$,

$$s(z) = \int_a^b \frac{\sigma(d\tau)}{\tau - z}, \tag{0.7}$$

called the associated function or Stieltjes transform of σ ($\sigma \in \mathcal{M}[a,b]$), is usually connected with the problem (0.4). Its asymptotic expansion

$$\begin{aligned} \int_a^b \frac{\sigma(d\tau)}{\tau - z} &= -\frac{1}{z} \int_a^b \left(1 - \frac{\tau}{z}\right)^{-1} \sigma(d\tau) \\ &= -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \int_a^b \tau^j \sigma(d\tau) \\ &= -\sum_{j=0}^{\infty} \frac{s_j^\sigma}{z^{j+1}}, \end{aligned} \tag{0.8}$$

reduces the considered moment problem to the problem of finding a set of σ such that $s_j^\sigma = s_j$, $j \in \{0, \ldots, k\}$. Here $s_j^\sigma = \int_a^b \tau^j \sigma(d\tau)$, $\sigma \in \mathcal{M}[a,b]$. That is, we find the set of measures $\sigma \in \mathcal{M}([a,b], \{s_j\}_{j=0}^k)$.
Let us remark that the Stieltjes transform determines the measure σ uniquely.

The *relation* between the problem (0.5) and (0.7) is given by the equation, (see [6])

$$\int_a^b \frac{\sigma(d\tau)}{z - \tau} = \frac{1}{z-a} \operatorname{Exp}\left(\frac{1}{L} \int_a^b \frac{f(\tau) d\tau}{z - \tau} \right). \tag{0.9}$$

The asymptotic expansion of the left and right sides of (0.9) gives

$$\begin{aligned} &\frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots \\ &\quad = \frac{1}{z-a} \operatorname{Exp}\left[\frac{1}{L} \left(\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots \right) \right]. \end{aligned} \tag{0.10}$$

The equality (0.10) turns into the following explicit relation between c_j and s_j, $j \in \{0, \cdots, k\}$ (see [7]), (here for simplicity, $a = 0$)

$$s_0 = 1, \ s_1 = \frac{c_0}{L}, \ s_2 = \frac{c_1}{L} + \frac{c_0^2}{2L^2},$$

$$s_3 = \frac{c_2}{L} + \frac{c_0 c_1}{L^2} + \frac{c_0^3}{6L^3},$$

$$\begin{aligned} s_j &= \frac{1}{j! L^j} \\ &\cdot \begin{vmatrix} c_0 & -L & \cdots & 0 \\ 2c_1 & c_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ j c_{j-1} & (j-1) c_{j-2} & \cdots & -jL \\ (j+1) c_j & j c_{j-1} & \cdots & c_0 \end{vmatrix} \\ &= \frac{c_j}{L} + \frac{c_0 c_{j-1}}{L^2} + \cdots, \quad (j \geq 1). \end{aligned} \tag{0.11}$$

Using the bijective relation (0.11), the L–Markov moment problem can be solved in terms of the $[a,b]$–Hausdorff moment problem. We realize the treatment of the last problem with help of the Potapov's FMI approach, (see [4], [5]). Let be remarked that in [4] and [5] an explicit solution of the nondegenerated matrix version of the problem Hausdorff moment problem was given.

Taking into account the remarkable difference in the construction of the solution of both cases, the even number and the odd number of data, we introduce first the matrices which appears in the FMI for even case (scalar version).

Definition 0.1 *Let $k = 2n+1$. Using the moments $s_0, s_1, \ldots, s_{2n+1}$ we construct the fol-*

lowing matrices

$$\begin{aligned}
\tilde{K}_1 &= \{s_{j+k}\}_{j,k=0}^{n},\ \tilde{K}_2 = \{s_{j+k+1}\}_{j,k=0}^{n} \\
T &= \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}}_{n+1}, \\
v &= column[1,0,\dots,0] \\
u &= column[-s_0,-s_1,\dots,-s_n], \\
R_T(z) &= (I-zT)^{-1}. \\
K_1 &= -a\tilde{K}_1+\tilde{K}_2, \quad K_2 = b\tilde{K}_1-\tilde{K}_2, \\
u_1 &= u-aTu, \quad u_2 = -u+bTu.
\end{aligned}$$

Further, we introduce two auxiliary holomorphic functions

$$\begin{aligned}
\tilde{s}_1(z) &= (z-a)s(z), \\
\tilde{s}_2(z) &= (b-z)s(z), z\in\mathbb{C}\setminus[a,b].
\end{aligned} \tag{0.12}$$

Where $s(z)$ is the Stieltjes transform of σ : $\sigma\in\mathcal{M}[a,b]$.

In a similar way we introduce the matrices for the Potapov's FMI odd case.

Definition 0.2 *Let $k=2n$. Let $T_1=T$, T is defined in definition (0.1). Using the moments $s_0,s_1,\dots,s_{2n}$ we construct the following matrices*

$$\begin{aligned}
K_1 &= \{s_{j+k}\}_{j,k=0}^{n}, \\
v_1 &= column[1,0,\dots,0], \\
u_1 &= column[0,-s_0,\dots,-s_{n-1}], \\
R_{T_1}(z) &= (I-zT_1)^{-1}, \\
\tilde{K}_1 &= \{s_{j+k+1}\}_{j,k=0}^{n-1}, \\
\tilde{K}_2 &= \{s_{j+k}\}_{j,k=0}^{n-1} \\
\tilde{K}_3 &= \{s_{j+k+2}\}_{j,k=0}^{n-1}, \\
K_2 &= (a+b)\tilde{K}_1-ab\tilde{K}_2-\tilde{K}_3, \\
T_2 &= \underbrace{\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}}_{n}, \\
v_2 &= column[1,0,\dots,0], \\
R_{T_2}(z) &= (I-zT_2)^{-1}, \\
\tilde{u}_1 &= column[-s_0,-s_1,\dots,-s_{n-1}], \\
\tilde{u}_2 &= column[0,-s_0,\dots,-s_{n-2}] \\
\tilde{u}_3 &= [-s_1,-s_2,\dots,-s_n], \\
u_2 &= (a+b)\tilde{u}_1-ab\tilde{u}_2-\tilde{u}_3.
\end{aligned}$$

Here $u_1,v_1\in\mathbb{R}^{n+1}$, $u_2,v_2\in\mathbb{R}^n$. I represents the unitary matrix of respective dimension. Further, we introduce two auxiliary holomorphic functions

$$\begin{aligned}
\tilde{s}_1(z) &= s(z), \\
\tilde{s}_2(z) &= (b-z)(z-a)s(z)-s_0z, \\
& \quad z\in\mathbb{C}\setminus[a,b].
\end{aligned} \tag{0.13}$$

Where $s(z)$ is the Stieltjes transform of a nonnegative σ on $[a,b]$.

We define the system of Potapov's FMI for the even and odd cases [4],[5].

Definition 0.3 *Let $(s_j)_{j=0}^{k}$ be a sequence of real numbers. The function s is called a solution of the associated system of V.P. Potapov's fundamental matrix inequality (FMI), if s satisfies the following properties:*

(i) *s is holomorphic in $\mathbb{C}\setminus[a,b]$.*

(ii) *For $r\in\{1,2\}$ the inequality*

$$\left[\begin{array}{c|c} K_r & R_{T_r}(z)\,[\,v_r\tilde{s}_r(z)-u_r\,] \\ \hline * & \{\tilde{s}_r(z)-\tilde{s}_r^*(z)\}/\{z-\bar{z}\} \end{array}\right] \geq 0 \tag{0.14}$$

holds.

Where K_r, T_r, u_r, $s_r(z)$ and v_r are defined as in (0.12) and (0.13). $*$ means the complex conjugate of $R_{T_r}(z)\,[\,v\tilde{s}_r(z)-u_r\,]$.

In definition 0.3 the auxiliary functions $\tilde{s}_r(z)$, $r\in\{1,2\}$ and T_k correspond to the even and odd cases of definition 0.1 and 0.2. Remark that for $k=2n+1$ the matrix $T_1=T_2=T$ and $v_1=v_2=v$.

It turns out that the treatment of the matrix moment problem is equivalent to finding all solutions of the system of corresponding fundamental matrix inequalities of Potapov type (see [4],[5]):

Theorem 0.1 *The function $s(z)$ is a Stieltjes transform of $\sigma:\sigma\in\mathcal{M}([a,b],\{s_j\}_{j=0}^{k})$ iff $s(z)$ is a solution of the system of Potapov's Fundamental Matrix Inequalities (0.14).*

This theorem takes place for both the even and the odd case of data. In this way the problem of finding the Stieltjes transform of σ is reduced to the problem of finding holomorphic functions $s(z)$ (see definition 0.3) such the inequality (0.14) holds.

Now we show that the TOC problem can be formulated in terms of a classical $[a, b]$–Hausdorff moment problem.

1 From TOC–problem to the classical moment problem.

The solution of the system (0.1) can be written in the following form:

$$x(\theta) = e^{A\theta}\left(x_0 + \int_0^\theta e^{-A\tau} b u(\tau) d\tau\right). \quad (1.1)$$

From the completely controllability of (0.1) there exists θ such that $x(\theta) = 0$.
Taking into account the relation

$$e^{-A\tau} b = \begin{pmatrix} 1 \\ -\tau \\ \vdots \\ \frac{(-1)^{n-1}}{(n-1)!}\tau^{n-1} \end{pmatrix}, \quad (1.2)$$

the equality (1.1) can be written in the form $-x_0^j = \frac{(-1)^{j-1}}{(j-1)!}\int_0^\theta \tau^{j-1} u(\tau) d\tau$, $j \in \{1, \ldots, n\}$.
We write the last relation in an equivalent form

$$\begin{aligned} (-1)^j (j-1)! x_0^j &= 2\int_0^\theta \tau^{j-1}\frac{(u(\tau)+1)}{2} d\tau \\ &\quad - \int_0^\theta \tau^{j-1} d\tau, \\ \frac{\theta^j + (-1)^j j! x_0^j}{2j} &= \int_0^\theta \tau^{j-1} f(\tau) d\tau, \\ & j \in \{1, \ldots, n\}. \quad (1.3) \end{aligned}$$

Thus, the TOC problem is reduced to the problem of finding the minimal θ and a function $0 \leq f(\tau) \leq 1$, $\tau \in [0, \theta]$ for which the relation (1.3) takes place.

Denote through $c_{j-1}(\theta, x_0)$, $j \in \{1, \ldots, n\}$ the left hand side of (1.3). Using the relation (0.11) for $L = 1$, $a = 0$, $b = \theta$, we obtain the data moments of the classical $[0, \theta]$–Hausdorff moment problem, which we symbolize through $s_j(\theta, x_0)$, $j \in \{0, \ldots, n\}$.

From the relation (0.11) we obtain the following

Preposition 1.1 *(See [7], pag. 324) The L–Markov moment problem with $c_{j-1}(\theta, x_0)$, $j \in \{1, \ldots, n\}$ entries is solvable iff the $[0, \theta]$–Hausdorff moment problem with entries $s_j(\theta, x_0)$, $j \in \{0, \ldots, n\}$ is solvable.*

In the next section we are going to show that the solution of the TOC problem is reduced to the solution of the Potapov's FMI (0.14) where the $\det K_1 = 0$ and/or $\det K_2 = 0$. In this way we obtain the optimal time θ and the points of switching of the optimal control $u(t)$.

2 Solution of the TOC problem.

Using the sequence $\{s_j(\theta, x_0)\}_{j=0}^n$, we construct Hankel matrices K_1, K_2 for even respectively odd numbers of data and vectors u_r, v_r, $r = \{1, 2\}$ as described in definition 0.1 and 0.2.
We find the optimal time $\theta(x_0)$ from the following preposition

Preposition 2.1 *The maximal real positive solution (root) of $\det K_1 = 0$ or/and $\det K_2 = 0$ (even and odd case) is the optimal time $\theta(x_0)$ of the system (0.1).*

The proof of the last preposition can be given in terms of the determinants of Markov (see [2]).
The condition $\det K_1 = 0$ or/and $\det K_2 = 0$ says that the considered $[0, \theta]$ Hausdorff moment problem is *degenerated*, consequently there is a unique solution.
Now we find the optimal control $u(t)$. Denote $\hat{K}$ the matrix K_r (even or odd case) such that $\det K_r = 0$, $r = 1$ or $r = 2$.
Denote $M = \begin{bmatrix} \nu & 0 \\ 0 & 1 \end{bmatrix}$, where $\nu \in \mathbb{R}^{n+1}$ or $\nu \in \mathbb{R}^n$, such that $\nu \in Ker\{\hat{K}\}$. Taking into

account the preposition 1.1 and theorem 0.1 consider the FMI (0.14). Write the inequality (0.14) in the equivalent form

$$M^*\left[\begin{array}{c|c}\hat{K} & R_{T_r}(z)\left[v_r\tilde{s}_r(z)-u_r\right] \\ \hline * & \{\tilde{s}_r(z)-\tilde{s}_r^*(z)\}/\{z-\bar{z}\}\end{array}\right]M$$

$$=\left[\begin{array}{c|c}\nu^*\hat{K}\nu & \nu^*R_{T_r}(z)\left[v_r\tilde{s}_r(z)-u_r\right] \\ \hline * & \{\tilde{s}_r(z)-\tilde{s}_r^*(z)\}/\{z-\bar{z}\}\end{array}\right]\geq 0 \tag{2.1}$$

Since $\nu^*\hat{K}\nu = 0$, from (2.1) we have $-|\nu^*R_{T_r}(z)\left[v_r\tilde{s}_r(z)-u_r\right]|^2 \geq 0$. Consequently $\nu^*R_{T_r}(z)\left[v_r\tilde{s}_r(z)-u_r\right]=0$. From the last equality we have

$$\tilde{s}_r(z)=\frac{\nu^*R_{T_r}(z)u_r}{\nu^*R_{T_r}(z)v_r}. \tag{2.2}$$

For the even case ($n=2p+1$) from (0.13) and (2.2) we have $s(z)=\dfrac{\nu^*R_T(z)u_1}{z\nu^*R_T(z)v}=\dfrac{P_1(z)}{Q_1(z)}$, or
$s(z)=\dfrac{\nu^*R_T(z)u_2}{(\theta_0-z)\nu^*R_T(z)v}=\dfrac{P_2(z)}{Q_2(z)}$
The condition $\det K_1=0$ or $\det K_2=0$ says which of the last expressions of s is to be taken.

Similarly for the odd case ($n=2p$) from (0.13) and (2.2) we have
$s(z)=\dfrac{\nu^*R_{T_1}(z)u_1}{\nu^*R_{T_1}(z)v_1}=\dfrac{P_3(z)}{Q_3(z)}$, or
$s(z)=\dfrac{\nu^*R_{T_2}(z)(u_2+s_0zv_2)}{(\theta_0-z)z\nu^*R_{T_2}(z)v_2}=\dfrac{P_4(z)}{Q_4(z)}$
Remark that $P_k, Q_k,\ k=\{1,\ldots,4\}$ are polynomials. Let $\dfrac{P(t)}{Q(t)}$ be the linear fractional relation corresponding to the condition $\det\hat{K}=0$. From the relation (0.9) we have $f(t)=\frac{1}{2}\left(1-sign\frac{P(t)}{Q(t)}\right)$. In terms of the control $u(t)$ we have, see [6]

$$u(t)=-sign\frac{P(t)}{Q(t)} \tag{2.3}$$

The switching points of (2.3) are given by the roots of $P(t)Q(t)$. The control (2.3) has not more than $(n-1)$ points of switching. By virtue of Lemma 9 of [2], the last control is the optimal control.
One the "advantages" of using the Potapov Method for solving the TOC problems is precisely the finding of the switching points of the optimal control without recursive operations.
Example. We find $\theta(x_0)$ and optimal control $u(t)$ for system (0.1), for $n=5$, $x_0=(0,0,0,0,x_5)$, $x_5>0$. See [2]. We have
$c_0=\dfrac{\theta}{2},\ c_1=\dfrac{\theta^2}{4},\ c_2=\dfrac{\theta^3}{6},\ c_3=\dfrac{\theta^4}{8},$
$c_4=\dfrac{-5!x_4+\theta^5}{10};$
$s_0=1,\ s_1=\dfrac{\theta}{2},\ s_2=\dfrac{3\theta^2}{8},\ s_3=\dfrac{5\theta^3}{16},$
$s_4=\dfrac{35\theta^4}{128},\ s_5=\dfrac{63\theta^5}{265}-12x_4.$
We use the even case Hausdorff scalar moment problem (see definition 0.1). The minimal time $\theta(x_0)=\theta_0$ is given by $\det K_1=0$ (its maximal real positive root). That is $\theta_0=4\sqrt[5]{6x_5}$. The vector ν such that, $\nu\in KerK_1$ is given by $\nu=column\left(\dfrac{5}{16}\theta_0^2\zeta,\ -\dfrac{5}{4}\theta_0\zeta,\ \zeta\right)$, $\zeta\in\mathbb{R}\setminus\{0\}$. The roots of the polynomials $P(t)=16t^2-12t\theta_0+\theta_0^2$ and $Q(t)=16t^2-20t\theta_0+5\theta_0^2$ give the switching points of control (2.3):
In the interval $\left[0,\dfrac{\theta_0}{8}(3-\sqrt{5})\right]$ the control $u(t)=-1$.
In $\left[\dfrac{\theta_0}{8}(3-\sqrt{5}),\dfrac{\theta_0}{8}(5-\sqrt{5})\right]$, $u(t)=+1$.
In $\left[\dfrac{\theta_0}{8}(5-\sqrt{5}),\dfrac{\theta_0}{8}(3+\sqrt{5})\right]$, $u(t)=-1$.
In $\left[\dfrac{\theta_0}{8}(3+\sqrt{5}),\dfrac{\theta_0}{8}(5+\sqrt{5})\right]$, $u(t)=+1$.
In $\left[\dfrac{\theta_0}{8}(5+\sqrt{5}),\theta_0\right]$, $u(t)=-1$.

Conclusion. The TOC problem was first reduced to a Markov moment problem under the additional condition $\theta\to\min$. Due to the relation (0.9), the last problem was reduced to Hausdorff moment problem on $[0,\theta]$. By virtue of theorem 0.1 the problem of finding the Stieltjes transform of solution (a measure σ) of the Hausdorff moment problem was "translated" to the problem of finding a solution (a holomorphic in $\mathbb{C}\setminus[a,b]$ function) of the FMI (2.1). We used the degenerated characteristic of the considered problem, that is, $\det K_1=0$ or/and $\det K_2=0$. These two last conditions guaranteed the unique solution of the TOC problem which was given with help of the V.P.Potapov's FMI method.

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