Characterizing families of Positive Real matrices by matrix-SPR0 substitutions on scalar rational functions

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Abstract

This paper concerns the characterization of families of (extended, strongly, and weak) positive real matrices generated by substitutions (of the Laplace variable s) in scalar rational transfer function by (extended, strongly, and weak, respectively) matrix positive real functions.

1 Introducción

As is pointed out in [16] and [17], the concept of *positive realness* of a transfer function plays a central role in *Stability Theory.* The definition of rational Positive Real functions (*PR* functions) arose in the context of *Circuit Theory.* In fact, the driving point impedance of a passive network is rational and positive real. If the network is *dissipative* (due to the presence of resistors), the driving point impedance of the network is a Strictly Positive Real transfer function (*SPR* function). Thus, positive real systems, also called *passive systems*, are systems that do not generate energy. The celebrated Kalman-Yakubovich-Popov (**KYP**) lemma (see for instance the Lefschetz-Kalman-Yakubovich version of this result in [16]), established the key role that strict positivity realness plays in Lyapunov functions associated to the stability analysis of a particular class of nonlinear systems, *i.e.*, Linear Time Invariant systems (**LTI** systems) with a single memory less nonlinearity. In fact, positive realness has been extensively studied by the Automatic Control community, see for instance the studies concerning:

- Synthesis of H_2 positive real controllers [10].
- Absolute stability [12].
- Characterization and construction of robust strict positive real systems [3], [20].
- Relationship between time domain and frequency domain conditions for strict positive realness [22], and
- Relationships between positivity realness of proper and stable **LTI** systems and stabilizing solutions of Riccati equations [23].
- Stability of adaptive control schemes based on parameter adaptation algorithms [14], [1].
- Passive filters [2].

As far as the frequency-described continuous **LTI** systems are concerned, the study of control-oriented properties (like stability) resulting from the substitution of the complex Laplace variable s by rational transfer functions have been little studied by the Automatic Control community. However, some interesting results have recently been published:

Concerning the study of the so-called *uniform* systems, *i.e.*, **LTI** systems consisting of identical components and amplifiers, it was established in [18] a general criterion for robust stability for functions of the form $D(f(s))$, where $D(s)$ is a polynomial and $f(s)$ is a rational transfer function. By applying such a criterion, it gave a generalization of the celebrated Kharitonov's theorem [13], as well as some robust stability criteria under H_∞-uncertainty. The results given in [18] are based on the so-called *H-domains*[1]. Unfortunately, obtaining *H-domains* is not an easy task (see for instance Lemma 1 in [18]).

As far as robust stability of polynomial families is concerned, some Kharitonov's like results [13] are

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[1] The *H-domain* of a function $f(s)$ is defined to be the set of points h on the complex plane for which the function $f(s) - h$ has no zeros on the open right-half complex plane.

given in [21] (for a particular class of polynomials), when interpreting substitutions as nonlinearly correlated perturbations on the coefficients.

More recently, in [4], some results for proper and stable real rational SISO functions and coprime factorizations were proved, by making substitutions with $\alpha(s) = (as+b)/(cs+d)$, where a, b, c, and d are strictly positive real numbers, and with $ad - bc \neq 0$. But these results are limited to the bilinear transforms, which are very restricted.

As far as robustness-oriented properties preservation in rational transfer function (modified by strictly positive real substitutions) are concerned, some results linked to H_∞-robustness are given in [8] for SISO (linear time-invariant) systems, while the MIMO case is presented in [9]. In [7] is given the characterization of families of algebraic Riccati equations associated to SISO systems bounded in H_∞-norm terms (the positive real substitutions acting on the bounded SISO systems). All these results have in common that they correspond to *scalar* positive real substitutions performed in *scalar* rational functions or/and *matrix* rational functions. In this paper we are mainly concerned by *matrix substitutions performed in scalar rational functions*. This topic is important since the complexity of the analysis of linear time-invariant systems, represented by matrix rational functions, can be reduced when working with scalar rational functions which affected by matrix substitutions would give rise to the matrix rational functions representing the analyzed systems (if it is the case). The computational complexity associated to control-oriented analysis (like the one concerning robust stability) would be reduced, and our results are just the first step in such direction.

The paper is organized as follows:

In Section 2 we give some preliminaries mainly concerning the notion of positive realness of rational functions, as well as some results concerning substitutions in the sacalr case.

In Section 3 we present some results concerning the preservation of properties in MIMO systems when performing substitutions of the Laplace variable by strictly proper positive substitutions of order equal to zero. We also recall the definition of extended, strongly and weak positive real functions.

We present our main results in Section 4, i.e., the preservation in matrix rational functions of the properties characterizing extended, strongly and weak positive realness, inherited from scalar functions in which the matrix substitutions are performed..

We conclude in Section 5 with some final comments.

2 Preliminaries

In this section, we give the notation, the basic definitions and some necessary results for the sequel.

Notation : $C^+ = \{\sigma + j\omega \in C \ : \ \sigma > 0\}$, $ImC \equiv \{z \in C \ : \ Re(z) = 0\}$,

$R =$ (-$\infty, \infty)$, $\overline{C}_e^+ \equiv C^+ \cup \{\infty\} \cup ImC$, $\overline{C}^+ \equiv C^+ \cup ImC$, $C_e^+ \equiv C^+ \cup \{\infty\}$.

Definition 1 *Let RC be the Euclidean domain of the proper, stable and rational real functions, $R(s)$ the field of real rational functions, $R_p(s)$ the ring of real rational and proper functions and $R[s]$ the ring of the real polinomials.*

Definition 2 *Let $p(s) \in R(s)$ be a rational function of complex variable $s = \sigma + j\omega$.*

1. *[12] $p(s)$ is Positive Real (PR) if:*

   (a) $p(s)$ is real for s real; (b) $Re[p(s)] \geq 0$ for all $Re[s] > 0$.

2. *[12] $p(s)$ is Strictly Positive Real (SPR) if $p(s$-$\varepsilon)$ is PR for some $\varepsilon > 0$.*

3. *[12], [5] $p(s)$ of zero relative degree is SPR (SPR0 function) if and only if:*

   (a) $p(s)$ is analytic in $Re[s] \geq 0$; (b) $Re[p(j\omega)] > 0$ for all $\omega \in R$.

4. *[15] $p(s)$ is Extended Strictly Positive Real (ESPR) if it is SPR and $Re[p(j\infty)] > 0$.*

5. *[15], [11] $p(s)$ is Strongly Strictly Positive Real (SSPR) if it is SPR and $Re[p(\infty)] > 0$.*

Definition 3 *[16] SPR0 $:= \{G(s) \in RH^\infty \mid G(s)\}$ is SPR0.*

Some properties of SPR0 functions are:

1. If $p(s)$ is a SPR0 function, then $1/p(s)$ is also a SPR0 function.
2. If $p_1(s)$ and $p_2(s)$ are SPR0 functions, then $\alpha p_1(s) + \beta p_2(s)$ is a SPR0 function for α, $\beta \geq 0$ (see [9]).

We can at this level present our:

Lemma 4 *[5] Consider a transfer function $p(s)$ be given.*

1. *If $p(s) \in RH^\infty$ with $q(s)$ any SPR0 function, then $p(q(s)) \in RH^\infty$.*

2. *If $p(s), q(s) \in$ SPR0, then $p(q(s)), q(p(s)) \in$ SPR0.*

3. *If the function $q(s) \in$ SPR0, then $q(\overline{C}_e^+) \subseteq C^+$.*

4. *If $p(s) \in$ SPR0, then $p(s)$ is ESPR.*

The following is a well-known result:

Lemma 5 *If $p(s) \in R(s)$, then:*

$$Re\,[p(j\omega)] = Re\,[p(-j\omega)]$$

and:

$$Im\,[p(j\omega)] = -Im\,[p(-j\omega)]$$

for all $\omega \in R$.

3 Results for MIMO systems

We present in this section some results concerning the preservation of real positivity properties in Multi-Input Multi-Output (MIMO) systems, when performing the substitution of the Laplace variable s by SPR0 functions.

Definition 6 *[12] A $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ is called Positive Real (PR) if:*

1. *All elements of $Z(s)$ are analytic for $Re[s] > 0$.*

2. *Any pure imaginary pole of any element of $Z(s)$ is a simple pole and the associated residue matrix of $Z(s)$ is positive semidefinite Hermitian, and:*

3. *For all real ω for which $j\omega$ is not a pole of any element of $Z(s)$, the matrix $Z(j\omega) + Z^T(\text{-}j\omega)$ is positive semidefinite ($Z(j\omega) + Z^T(\text{-}j\omega) \geq 0$).*

Remark 1 *Again, a $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ is called Strictly Positive Real (SPR), if $Z(s\text{-}\varepsilon)$ is PR for some $\varepsilon > 0$. Note also, that if $Z(s)$ is SPR, then there exist some $\varepsilon > 0$ such that $Z(s\text{-}\varepsilon)$ is PR.*

Lemma 7 *Let $Z(s)$ be a $\rho\times\rho$ proper rational transfer function matrix, and suppose $\det\left[Z(s)+\ Z^T(-s)\right]$ is not identically zero. Then, $Z(s)$ is SPR if and only if:*

1. *$Z(s)$ is Hurwitz i.e., $Z(s) \in \mathbf{RH}^\infty$ where $\mathbf{RH}^\infty$ is the set of matrices with elements in RH^∞,*

2. *$Z(j\omega) + Z^T(-j\omega) > 0$ for all real ω, and*

3. *one of the following three conditions is satisfied:*

   (a) *$Z(\infty) + Z^T(\infty) > 0$;*

   (b) *$Z(\infty) + Z^T(\infty) = 0$ and $\lim_{\omega\to\infty} \omega^2 \left[Z(j\omega) + Z^T(-j\omega)\right] > 0$;*

   (c) *$Z(\infty) + Z^T(\infty) \geq 0$ and there exist positive constants σ_0 and ω_0 such that*

$$\omega^2 \sigma_{\min}\left[Z(j\omega) + Z^T(-j\omega)\right] \geq \sigma_0,\ \forall\,|\omega| \geq \omega_0.$$

Definition 8 *A $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ is called Extended Strictly Positive Real (ESPR) if it is SPR and $Z(j\infty) + Z^T(\text{-}j\infty) > 0$.*

Definition 9 *[15] Let $Z(s)$ be a $\rho\times\rho$ proper rational transfer function matrix. Then:*

1. *$Z(s)$ is called strongly SPR (SSPR), if $Z(s)$ is SPR and $Z(\infty) + Z^T(\infty) > 0$.*

2. *$Z(s)$ is called weak SPR (WSPR), if $Z(s)$ is SPR and: $Z(j\omega) + Z^T(\text{-}j\omega) > 0$ for all $\omega \in R$.*

3. *$Z(s)$ is called MSPR, if $Z(s)$ is PR and $Z(j\omega) + Z^T(\text{-}j\omega) > 0$ for all $\omega \in R$.*

The following result is evident:

Lemma 10 *Let $Z(s)$ be a $\rho\times\rho$ proper rational transfer function matrix. then:*

1. *If $Z(s)$ is a SPR function matrix, then $Z(s + \varepsilon)$ is a SPR $\rho \times \rho$ proper rational transfer function matrix, for each $\varepsilon \geq 0$.*

2. *If $Z(s)$ is a PR function matrix, then $Z(s + \varepsilon)$ is a SPR function matrix for each $\varepsilon > 0$.*

We can at this level present our:

Theorem 11 *Consider a transfer function matrix $Z(s) \in \mathbf{RH}^\infty$ be given.*

1. *If $Z(s) \in \mathbf{RH}^\infty$, then $Z(p(s)) \in \mathbf{RH}^\infty$for each $p(s) \in$SPR0,*

2. *if $Z(s)$ is a SPR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$ is a ESPR $\rho \times \rho$ proper rational transfer function matrix, for each $p(s) \in$ SPR0.*

3. *If $Z(s)$ is a PR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$is a ESPR $\rho \times \rho$ proper rational transfer function matrix, for each $p(s) \in$ SPR0.*

Please remark that by the definition of SSPR, if $Z(s)$ is either SPR or PR, then $Z(p(s))$ is SSPR for each $p(s) \in$ SPR0, too. In what follows we apply the results corresponding to Theorem 11 to the matrix function classes introduced in Definition 8.

Definition 12 *Let $Z(s)$ be a $\rho \times \rho$ proper rational transfer function matrix.*

1. *[11] $Z(s)$ is called bounded real (BR) if:*
   (a) *All elements of $Z(s)$ are analytic for $Re[s] \geq 0$, and:*
   (b) *I - $Z^T(-j\omega)Z(j\omega) \geq 0$ for all $\omega \in R$. Equivalently, the condition b) can be replaced by:*
   (c) $\|Z(s)\|_\infty \leq 1$.
2. *[11] $Z(s)$ is called strictly bounded real (SBR) if:*
   (a) *All elements of $Z(s)$ are analytic for $Re[s] \geq 0$, and:*
   (b) *I - $Z^T(-j\omega)Z(j\omega) > 0$ for all $\omega \in R$. Again, the condition b) can be replaced by:*
   (c) $\|Z(s)\|_\infty < 1$.

Remark 2 *If the transfer function matrix $Z(s)$ is SBR, then I - $D^T D > 0$, where $D := Z(\infty)$.*

The following corollary follows:

Corollary 13 *If $Z(s)$ is ESPR, SSPR, WSPR or MSPR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$ is ESPR and SSPR $\rho \times \rho$ proper rational transfer function matrix, for each $p(s) \in$SPR0.*

Lemma 14 *[1], [19] Consider a $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ be given. Then:*

1. *Suppose that $Z(s)$ satisfies $\det(Z(s) + I) \neq 0$ for $Re[s] \geq 0$, then $Z(s)$ is ESPR if and only if:*
$$H(s) = (I - Z(s))(I + Z(s))^{-1}$$
   is SBR. Also, $Z(s)$ is PR if and only if $H(s)$ is BR.

   Equivalently, item 1 can be replaced by:
2. *Consider a $\rho \times \rho$ proper rational transfer function matrix $H(s)$ satisfying $\det(I + H(s)) \neq 0$, for $Re[s] \geq 0$. Then, $H(s)$ is SBR if and only if:*
$$Z(s) = (I + H(s))^{-1}(I - H(s))$$
   is ESPR. Also, $H(s)$ is BR if and only if $Z(s)$ is PR.

Proposition 15 *Suppose that $Z(s)$ and $H(s)$ are $\rho \times \rho$ proper rational transfer function matrices, such that $\det(Z(s) + I) \neq 0$ and $\det(I + H(s)) \neq 0$ for $Re[s] \geq 0$, then:*

1. *If $Z(s)$ is PR, SPR, ESPR, SSPR, WSPR or MSPR $\rho \times \rho$ proper rational transfer function matrix, then $H(p(s))$ is SBR for each $p(s) \in$SPR0.*
2. *If $H(s)$ is either SBR or BR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$ is ESPR for each $p(s) \in$SPR0.*
3. *If $H(s)$ is either SBR or BR $\rho \times \rho$ proper rational transfer function matrix, then $H(p(s))$ is SBR for each $p(s) \in$SPR0.*
4. *If $G(s) \in \mathbf{RH}^\infty$ is such that either $\|G(s)\|_\infty < \gamma$ or $\|G(s)\|_\infty \leq \gamma$, then $\|G(p(s))\|_\infty < \gamma$ for each $p(s) \in$SPR0.*

4 Main results

We present is this section our main results. First of all we establishe that the rational matrix resulting from the composition of a scalar proper rational function with a proper rational and stable matrix is also proper rational and stable, if its determinant is a unit in the space of proper rational and stable matrices. Finally, we establishe the preservation of (some particular) positive realness properties when performing matrix positive real substitutions on scalar positive real functions.

Proposition 16 *If $g(s) = \frac{p(s)}{q(s)}$ where $p(s)$, $q(s) \in R[s]$ (and $g(s)$ proper), $Z(s) \in \mathbf{RH}^\infty$ and $\det q(Z(s))$ is a unit in RH^∞. Then $g(Z(s)) \in \mathbf{RH}^\infty$.*

We can present now our:

Theorem 17 *If the following three conditions are satisfy:*

1. *$\det f(Z(s))$ is a unit in RH^∞.*
2. *$f(s) \in$SPR0.*
3. *$\det\left[Z(s) + Z^T(-s)\right]$ is not identically zero.*

   Then if $Z(s)$ is a SSPR, ESPR, WSPR or MSPR, $\rho \times \rho$ proper rational transfer function matrix, then $f(Z(s))$ is SSPR, ESPR, WSPR or MSPR, $\rho \times \rho$ proper rational transfer function matrix, respectively.

5 Final comments

We presented in this paper an study concerning the preservation of the properties characterizing extended, strongly, and weak positive realness in the scalar case when performing matrix substitutions (see

Theorem 17). Which is to say, we took scalar rational functions belonging to the (extended, strongly, and weak) positive real families and we substitute the Laplace variable s by matrix functions also characterized by the positive realness properties of the corresponding scalar function.

Some interesting problems associated to substitutions are interesting and should be addressed. In particular the following one is very appealing:

If a matrix rational function belongs to one of the specified families, is this matrix the result of a matrix substitution performed on a scalar rational function? Even in the scalar case this problem remains open (see [6]).

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